UDC 539.3

PLANE PROBLEM OF THE THEORY OF ELASTICITY IN MULTIPLY-CONNECTED DOMAINS WITH CYCLIC AND MIRROR SYMMETRY

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The first boundary value problem of the theory of elasticity is studied for the regions and loads possessing not only the cyclic [1, 2], but also the mirror symmetry. The form of the integral Sherman — Lauricella equation is affected by the type of symmetry.

1. Let a region D, in general multiply-connected, be defined on a plane with a rectangular x, y-coordinate system. We shall consider the regions which have mirror symmetry with respect to the rays $\varphi = \pi r / n$ $(r = 0, 1, \ldots, 2n - 1)$ of the polar $R\varphi$-coordinate system. Obviously these regions will also have a rotational symmetry about the coordinate center by the angles $\alpha_k = 2\pi k / n$ $(k = 1, 2, \ldots, n - 1)$. The region D is bounded by several closed contours: $L_0, L_1, \ldots, L_m$, and the contour L_0 contains all the remaining contours. We denote the sum of all boundaries by L. Known external loads are applied along these contours, and the loads possess the same symmetry as the region D. We assume that the principal vectors of the external loads applied to the contours L_k $(k = 0, 1, \ldots, m)$ are equal to zero.

As we know, [3, 4] the above problem reduces that of finding two functions $\varphi(z)$ and $\psi(z)$ of the complex variable z, single-valued in D, the functions satisfying the following conditions at the boundaries:

$$\varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)} = f(t) + C_j \text{ on } L_j \tag{1.1}$$

where $f(t)$ is a function defined in terms of the external loads and $C_0, C_1, \ldots, C_m$ are unknown complex constants one of which can be fixed by putting $C_0 = 0$.

From the conditions of mirror symmetry about the x-axis follows

$$\varphi(z) = \overline{\varphi(\bar z)}, \quad \psi(z) = \overline{\psi(\bar z)} \tag{1.2}$$

The symmetry of rotation by the angle $\alpha_k = 2\pi k/n$ holds when [2]

$$\varphi(z) = \exp(i\alpha_k)\, \varphi(z \exp(-i\alpha_k)), \quad \psi(z) = \exp(-i\alpha_k)\, \psi(z \exp(-i\alpha_k)) \tag{1.3}$$

Following Sherman [3] we shall seek the functions $\varphi(z)$ and $\psi(z)$ in the form

$$\varphi(z) = \frac{1}{2\pi i} \int_L \frac{\omega(t)\, dt}{t - z} \tag{1.4}$$

$$\psi(z) = \frac{1}{2\pi i} \int_L \left\{ \frac{\overline{\omega(t)}\, dt + \omega(t)\, d\bar t}{t - z} - \frac{\bar t\, \omega(t)\, dt}{(t - z)^2} \right\} + \sum_{j=1}^{m} \frac{b_j}{z - z_j} \tag{1.5}$$

$$b_j = i \int_{L_j} \{\omega(t)\, d\bar t - \overline{\omega(t)}\, dt\}, \quad j = 1, \ldots, m \tag{1.6}$$

Here $\omega(t)$ is a function of the points of the contour and is to be determined, z_j denote the points within the regions bounded by the contours L_j $(j = 1, \ldots, m)$, and the points are chosen such that for symmetrical contours the points have the same symmetry, and b_j are real constants which can be determined from the above formulas. Integration along the contour L_0 is carried out in the anticlockwise direction and along the remaining contours in the clockwise direction.

The expression (1.4) for $\varphi(z)$ usually includes another term identical with the last term of (1.5). The term has no significant meaning (see [5]) and the solution can be sought in the form (1.4). However, for the problems possessing a symmetry it is important that $\varphi(z)$ is represented precisely by (1.4) since the function $\omega(t)$ has, in this case, the same simple symmetry properties as the function $\varphi(z)$, and the derivation of the integral equation for the symmetrical problems no longer presents any difficulties, in constrast to [1, 2], where a different representation of $\varphi(z)$ was used.

The symmetry conditions (1.2) and (1.3) will hold, if the following relations hold:

$$\omega(t) = \overline{\omega(\bar{t})}, \quad \omega(t) = \exp(i\alpha_k)\,\omega(t\exp(-i\alpha_k)) \tag{1.7}$$

2. Let us denote by Γ_{p0}, $p = 0, 1, \ldots, M$, the parts of the contours $L_0, L_1, \ldots, L_m$ lying within the angle $0 \leqslant \varphi \leqslant \pi/n$, and let Γ_{00} denote a part of the outermost contour. Let Γ_{p0}^* be a mirror reflection of Γ_{p0} in the x-axis, and Γ_{pk} or Γ_{pk}^* the contour obtained by rotating Γ_{p0} or Γ_{p0}^* by the angle α_k. Clearly, the boundary of the region D is a sum of the contours $\Gamma_{pk} + \Gamma_{pk}^*$ with k varying from 0 to $n-1$.

Using the above notation and (1.6), we can write (1.1), after substituting into it (1.4) and (1.5), in the form

$$\omega(t_0) + \frac{1}{2\pi i}\sum_{k=0}^{n-1}\sum_{p=0}^{M}\int\limits_{\Gamma_{pk}+\Gamma_{pk}^*}\left\{\omega(t)\,d\ln\frac{t-t_0}{\bar{t}-\bar{t}_0} - \omega(t)\,d\,\frac{t-t_0}{\bar{t}-\bar{t}_0}\right\} + \tag{2.1}$$

$$i\sum_{k=0}^{n-1}\sum_{p=0}^{M}\left\{\int\limits_{\Gamma_{pk}}\frac{\omega(t)\,d\bar{t} - \overline{\omega(t)}\,dt}{\bar{t}_0 - \bar{z}_{pk}} + \int\limits_{\Gamma_{pk}^*}\frac{\omega(t)\,d\bar{t} - \overline{\omega(t)}\,dt}{\bar{t}_0 - z_{pk}}\right\} -$$

$$C_l = f(t_0), \quad t_0 \in \Gamma_{l0}$$

where z_{pk} denotes the point z_j of (1.5) corresponding to the contour Γ_{pk}. By definition, $z_{pk} = z_{p0}\exp(i\alpha_k)$.

Let us change in the integrals along the contours $\Gamma_{pk}, \Gamma_{pk}^*$ $(k = 1, \ldots, n-1)$ the variable of integration $\tau = t\exp(-i\alpha_k)$. According to the second equation of (1.7) we have for $t \in \Gamma_{pk}$, $t \in \Gamma_{pk}^*$, $\omega(t) = \omega(\tau)\exp(i\alpha_k)$. This transforms the contour Γ_{pk} into Γ_{p0} and Γ_{pk}^* into Γ_{p0}^*. Changing now the variable τ in the integrals along the contours Γ_{p0}^*, $\tau = \bar{t}$, $\omega(\tau) = \overline{\omega(\bar{\tau})} = \overline{\omega(t)}$ we find, that L_{p0}^* becomes L_{p0} but with the opposite direction. Taking this into account, we obtain

$$\omega(t_0) + \frac{1}{2\pi i}\sum_{k=0}^{n-1}\sum_{p=0}^{M}\int\limits_{\Gamma_{p0}}\{\omega(t)\exp(i\alpha_k)\,d\ln\xi_k(t) - \tag{2.2}$$

$$\overline{\omega(t)}\exp(i\alpha_k)\,d\ln\xi_k(\bar{t}) - \overline{\omega(t)}\exp(-i\alpha_k)\,d\xi_k(t) +$$

$$\omega(t)\exp(-i\alpha_k)\,d\xi_k(\bar{t})\} + i\sum_{k=0}^{n-1}\sum_{p=1}^{M}\int_{\Gamma_{p0}}\left\{\frac{\omega(t)\,d\bar{t} - \overline{\omega(t)}\,dt}{\bar{t}_0 - \bar{z}_{p0}\exp(-i\alpha_k)} + \right.$$

$$\left.\frac{\omega(t)\,d\bar{t} - \overline{\omega(t)}\,dt}{\bar{t}_0 - z_{p0}\exp(i\alpha_k)}\right\} - C_l = f(t_0)$$

$$\xi_k(t) = \frac{t\exp(i\alpha_k) - t_0}{\bar{t}\exp(-i\alpha_k) - \bar{t}_0} \tag{2.3}$$

Let us determine C_l. If Γ_{l0} is a closed contour, then according to Sherman we have

$$C_l = -\int_{\Gamma_{l_0}} \omega(t)\,ds$$

Consider the case in which Γ_{l0} is an open contour. Let Γ_{l0} and Γ_{l0}^* form a closed contour intersecting the x-axis. Then, using the condition of mirror symmetry expressed by the first equation of (1. 7), we obtain

$$C_l = -\int_{\Gamma_{l_0}} [\omega(t) + \overline{\omega(t)}]\,ds$$

Let the contour Γ_{l0} touch the ray $\varphi = \pi / n$. The condition of mirror symmetry with respect to this ray yields

$$\omega(t) = \omega(\bar{t}\exp(2i\pi/n))\exp(2i\pi/n)$$

and this enables us to write C_l in the form

$$C_l = -\int_{\Gamma_{l_0}}\left[\omega(t) + \overline{\omega(t)}\exp\left(2i\,\frac{\pi}{n}\right)\right]ds$$

Finally, we consider the case when Γ_{l0} represents a part of the closed contour enclosing the coordinate origin. Using the rotational symmetry by the angle α_k, we obtain

$$C_l = -\sum_{k=0}^{n-1}\left\{\int_{\Gamma_{l_0}}\omega(t)\exp(i\alpha_k)\,ds + \int_{\Gamma_{l_0}^*}\omega(t)\exp(i\alpha_k)\,ds\right\}$$

But for $n \geqslant 2$ we have

$$\sum_{k=0}^{n-1}\exp\left(ik\,\frac{2\pi}{n}\right) = \frac{1-\exp(2\pi i)}{1-\exp(2\pi i/n)} = 0$$

Consequently, if a rotational symmetry by an angle smaller than 2π exists, then $C_l = 0$ for the contour in question.

Denote by Γ the sum of all contours Γ_{p0}, and introduce on Γ a piecewise constant function $p(t)$: $p(t) = p$ if $t \in \Gamma_{p0}$, i. e. $p(t)$ assumes the value of the number of the contour containing the point t. Now we can write equation (2. 2) in the form

$$\omega(t_0) + \frac{1}{2\pi i}\int_{\Gamma}\sum_{k=0}^{n-1}\left\{\exp(i\alpha_k)\,[\omega(t)\,d\ln\xi_k(t) - \overline{\omega(t)}\,d\ln\xi_k(\bar{t})] - \right. \tag{2.4}$$

$$\exp(-i\alpha_k)\,[\overline{\omega(t)}\,d\xi_k(t) - \omega(t)\,d\xi_k(\bar{t})] -$$

$$2\pi\left[\omega(t)\,d\bar{t}-\overline{\omega(t)}\,dt\right]\left[\frac{1}{\bar{t}_0-\bar{z}_p\exp(-i\alpha_k)}+\frac{1}{\bar{t}_0-z_p\exp(i\alpha_k)}\right]\Big\}-$$
$$C_l=f(t_0),\quad t_0\in\Gamma_{l0},\quad l=p(t_0),\quad p=p(t),\quad z_p=z_{p0}$$

where $\xi_k(t)$ is given by (2.3).

Thus the function $\omega(t)$ must satisfy equation (2.4) defined on the part Γ of the contour L of the region D. The solvability of (2.4) follows from the solvability of the Sherman — Lauricella equation. It is true that in order to prove the uniqueness of the solution of the latter equation Sherman introduced into it another term [4] with an unknown imaginary multiplier b_{m+1}. In the presence of mirror symmetry however, this term vanishes by virtue of the first equation of (1.7).

It should be noted that for the regions with cyclic symmetry only, equation (2.4) must contain an additional term as in [1, 2]. But even then the function will satisfy the second equation of (1.7).

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